Hausdorff's measure and fractal objects

Takla Nateeboon

April 13, 2024

Abstract

One of the interesting properties of fractals is that some fractals may have infinitely large perimeters. The notion of length and measurements based on length is then not suitable to quantitatively describe the fractal. Here, we review a construction of the Hausdorff dimension from the Hausdorff measure. The review starts with Caratheodory's construction of the Hausdorff measure and its relation to the Lebesgue measure. Once the measure is defined, the definition of the Hausdorff dimension is introduced. We calculated the dimensions of a few fractal and non-fractal objects. Lastly, we show its relation with the similarity dimension.

Contents

1 Motivation

1.1 Coastline paradox and the study of fractals

The concepts of counting, length, area, and volume are useful quantities that allow us to describe many physical objects. However, some objects cannot be properly described through these quantities. The most famous example of this is a coastline. Measuring the length of a coastline on a map of scale 1:100,000 will result in a different length when measuring from a map with a scale of 1:1,000. The difference is not in terms of how precisely one can measure the coastline. As we measure more and more precisely, the coastline length becomes longer. This is because the roughness of the coastline presents in every scale[1]. As one measures with more accuracy, one adds finer peninsulas and bays to the length. The difference is firstly observed from Richardson. The ad-hoc solution people of that day proposed was to agree on the scale of measurement. Later Mandelbrot [Man82], pointed out in his essay that one can use a measure proposed decades earlier by Hausdorff to properly describe these objects.

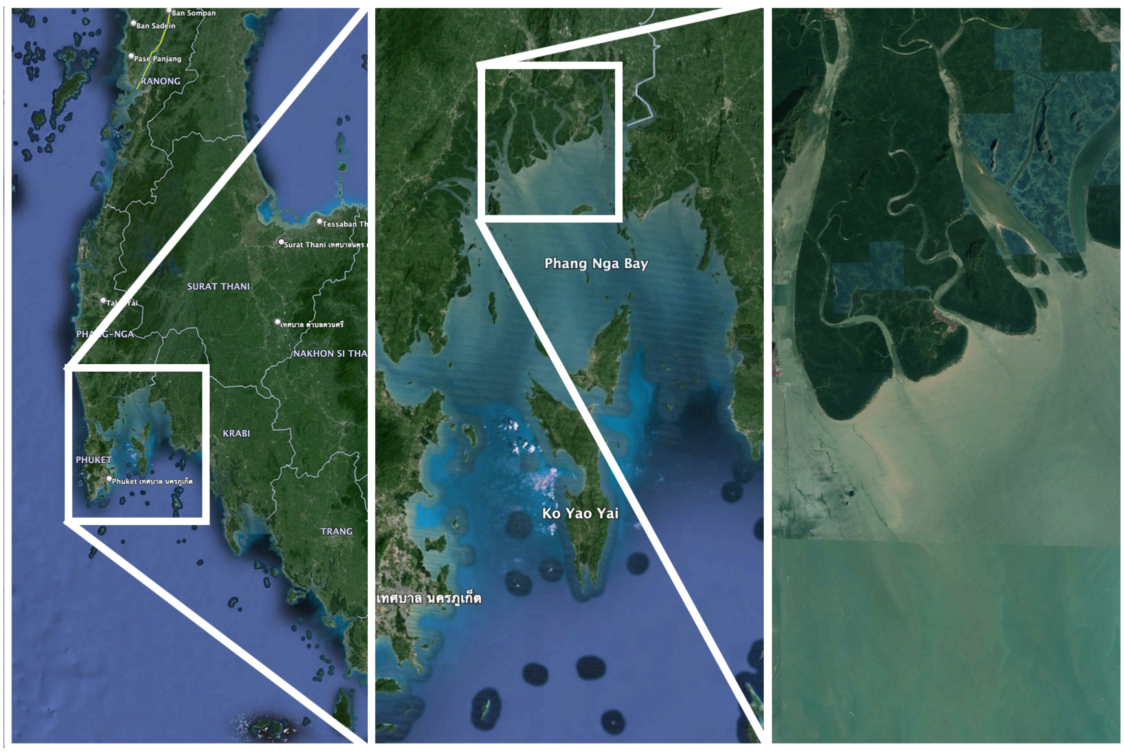


Figure 1: Satellite images of the western coastline of Thailand. These are screenshots from Google Earth. As one inspects the image with more precision, from left to right, one can see hidden details. The length of the coastline has to take these finer details into account.

The coastline is one of the examples of natural fractals. A mathematical fractal is more idealized than a natural fractal. The natural fractal is guaranteed to stop having the roughness once reaches the atomic level, while the mathematical one is constructed (e.g. Koch curve) or discovered (Mandelbrot set) to be infinitesimally rough.

[1]Mandelbrot stated that the roughness vanishes at the scale of 50 cm. Even if this is true, measuring an entire coastline of tens of thousands of kilometers in length to the precision of 50 cm is tedious work.

For an example of a fractal, consider the Koch curve. It is easy to construct along. Once constructed, one can be convinced that its perimeter is infinitely long. We show how a Koch curve is constructed in the figure 2. A Koch curve is a subset of $\mathbb{R}^2$ which is a limit of the following construction.

1. Start with a line of finite length, says $[0, 1]$.
2. Divide the line into three parts with equal length
3. Replace the middle part with an equilateral triangle that has no base
4. Repeat the second step for each line segment

Figure 2: Koch curve construction. The figure shows the first 4 fractals of the Koch curve.

At each step, the length of 1/3 is added to the curve, while the curve starting point and end point are zero and one. The length of the curve is not bounded.

The Koch curve has a property which is called self-similar, meaning that there is a smaller part of the Koch curve that looks similar to the whole Koch curve. However, fractals need not be self-similar and some self-similar objects are not fractal. To be precise, Mandelbrot defined a fractal to be an object with a Hausdorff dimension greater than its topological dimension; independent of self-similarity.

1.2 Generalization of Lesbesgue measure to non-integer dimension

For one-dimensional objects, the measure of length may be finite while the area and volume of them are none. For a fractal curve, the length of a curve can be infinite as in the example of the coastline and Koch curve, while the area of the curve is none. There is a missing step between a measure of infinite and a measure that results in none. To tame this infinity, one can look at the Hausdorff measure to quantitatively say how much substance is in the set corresponding to this type of objects.

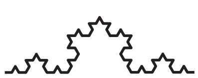
	Lenght	**Area**	**Volumn**
Line segment	finite	0	0
 Rectangle	finite, ∞	finite	0
 Koch curve	∞	0	0

Figure 3: Line segment and rectangle both can be described with length or area, however, we don't have much information on how much substance is in the Koch curve when saying the Koch curve has infinite length and zero area.

2 Hausdorff measure

2.1 Mathematical Construction of the measure

Given a separable metric space (X, d), we can define $\mathcal{H}^r$ a Hausdorff's r-dimensional measure using Caratheodory's construction. The construction here follows mostly from Chapter 12 of [Tay06]. Intuitively, this is analog to the measure of a coastline length with a meter stick of length no greater than δ.

Definition 2.1 (Hausdorff's set function).

$$h^*_{r,\delta}(S) = \inf\left\{\sum_{j\geq 1}(\text{diam } B_j)^r : S \subseteq \bigcup_{j\geq 1} B_j, \text{diam } B_j \leq \delta\right\} \tag{1}$$

using a convention $\inf \emptyset = \infty$ and $\text{diam } B_j = \sup\{d(x,y) : x, y \in B_j\}$

The limit of this set function is a metric outer measure.

Proposition 2.2. $h^*_r = \lim_{\delta\to 0} h^*_{r,\delta}$ is a metric outer measure.

Proof. There are two things to prove: h^*_r is an outer measure and h^*_r is a metric outer measure.

First, to show that h^*_r is an outer measure, we have to show the following properties:

(a) $h^*_r(\emptyset) = 0$

(b) If $A \subseteq B$, then $h^*_r(A) \leq h^*_r(B)$.

(c) $h^*_r\left(\bigcup_{j\geq 1} A_j\right) \leq \sum_{j\geq 1} h^*_r(A_j)$ for any countable collection $\{A_j\}_{j\geq 1}$.

(a) is straightforward as $\emptyset \subseteq \{x\}$ for an $x \in X$ and the diameter of a singleton is 0. For (b), if B contains A then the diameter of B is at least the diameter of A. The covering sets of B cover the set A also. Then, the sum of the diameter to the power of r of covering sets of B is at least as great as that of A. The proof of (c) is studied from [Nat12]. For (c), for each A_j choose a covering set B_{jk} such that

1. $A_j \subseteq \bigcup_{k\geq 1} B_{jk}$,
2. $\text{diam } B_{jk} < \delta$, and
3. for all $\varepsilon > 0$, $\sum_{k\geq 1}\left(\text{diam } B^r_{jk}\right) < h^*_r(A_j) + \frac{\varepsilon}{2^i}$.

Once the covering set is chosen, we have

$$\bigcup_{j\geq 1} A_j \subseteq \bigcup_{j\geq 1}\bigcup_{k\geq 1} B_{jk}.$$

The Hausdorff set function $h^*_{r,\delta}$ cannot be greater than the sum of B_{jk}'s diameter over j, k, i.e.

$$h^*_{r,\delta}\left(\bigcup_{j\geq 1} A_j\right) \leq \sum_{j,k\geq 1}(\text{diam } B_{jk})^r < \sum_{j\geq 1} h^*_r(A_j) + \frac{\varepsilon}{2^i}$$

As this hold for all $\varepsilon > 0$ and take limit $\delta \to 0$, we must have

$$h^*_r\left(\bigcup_{j\geq 1} A_j\right) \leq \sum_{j\geq 1} h^*_r(A_j).$$

Secondly, to show that h_r^* is a metric outer measure, we will show that if the distance between any two sets A and B is non-zero, then the outer measure is additive under the union of the two sets. Suppose the distance between the two sets, $d(A,B)$, is

$$d(A,B) = \inf\ \{d(x,y) : x \in A,\ y \in B\} = \varepsilon > 0. \tag{2}$$

Since the distance between two sets is finite, when the δ shrinks, the covering sets, C_j, may intersect with either A or B, but not both. Consider $h_{r,\delta}^*$ when $\delta < \varepsilon/2$,

$$h_{r,\delta<\varepsilon}^* = \inf \left\{ \sum_{j\geq 1} (\text{diam } C_j)^r :\ A \cup B \subseteq \bigcup_{j\geq 1} C_j,\ \text{diam } C_j \leq \delta < \varepsilon/2 \right\} \tag{3}$$

The collection of covering sets can be split into three groups: those intersecting A but not B, vice versa, and those not intersecting with both A and B. Since the diameter is non-negative, the latter group can be removed as it can only increase the sum of the diameters. The expression of the union of all covering sets becomes

$$\bigcup_{j\geq 1} C_j = \bigcup_{k:C_k\cap A\neq\emptyset} C_k \cup \bigcup_{\ell:C_\ell\cap B\neq\emptyset} C_\ell. \tag{4}$$

Moreover, the set in the first group never intersects the second group as $\delta < \varepsilon/2$. Then the infimum of the whole $A \cup B$ is the infimum of those covering A adding with the infimum of those covering B. This properties hold for all $\delta < \varepsilon/2$, then its limit, if exists, also has this property. □

Since h_r^* is a metric outer measure of (X), every Borel subset of X is measurable. Then, we define the Hausdroff r-dimensional measure of a Borel set A by

Definition 2.3. Hausdroff r-dimensional measure A Hausdroff r-dimensional measure of a Borel subset A of X, $\mathcal{H}^r(A)$, is defined as

$$\mathcal{H}^r(A) = \gamma_r h_r^*(A) \tag{5}$$

when γ_r is

$$\gamma_r = \frac{\pi^{r/2} 2^{-r}}{\Gamma\left(\frac{r}{2}+1\right)} \tag{6}$$

Notice that for a hypervolume of the hypersphere of diameter d is

$$\frac{\pi^{r/2} 2^{-r}}{\Gamma\left(\frac{r}{2}+1\right)} d^r.$$

For the derivation of the hypervolume see this online encyclopedia [Wei] on Wolfram Mathworld. The rationale behind multiplication with γ_r is to make the Hausdorff measure coincide with the Lebesgue measure in $\mathbb{R}^r$ when r is a whole number. For example, at $r = 1$ we have

$$\gamma_1 = \frac{\sqrt{\pi} 2^{-1}}{\Gamma(\frac{1}{2}+1)} = \frac{\sqrt{\pi} 2^{-1}}{\frac{2!}{4}\sqrt{\pi}} = 1.$$

The Hausdorff 1-dimensional measure is

$$\mathcal{H}^1(A) = h_1^*(A) = \lim_{\delta\to 0} \inf \left\{ \sum_{j\geq 1} \text{diam } B_j : A \subseteq \bigcup_{j\geq 1} B_j, \text{diam } B_j \leq \delta \right\}, \tag{7}$$

which is equivalent to the Lebesgue measure in one-dimensional space.

2.2 Relation to the Lebesgue measures

As aforementioned, the Hausdorff n-dimensional measure coincides with the Lebesgue measure of the same dimension when n is an integer. Most of the theorems and their proof in this section are studied from Chapter 2 of [EG15].

Theorem 2.4. Suppose A is a Borel subset of $\mathbb{R}^n$, then

$$\mathcal{L}^n(A) = \mathcal{H}^n(A). \tag{8}$$

Before we proceed to the proof of the theorem, the definition of n-dimensional Lebesgue measure is a product measure of the Lebesgue measure in 1 dimension. The theorem says that the Lebesgue measure which is derived from the covering of $n-$dimensional cubes numerically agrees with the Hausdorff measure which is given by arbitrary $n-$dimensional shape but infinitesimally small shapes.

Proof of theorem 2.4. Firstly, by isodiametric inequality (see Lemma 2.8 and its proof) we have $\mathcal{L}^n(A) \leq \sum_{j=1}^{\infty} \gamma_n(\text{diam } C_j)^n$ where $\{C_j\}$ is any collection of covering sets for A. Taking the infimum of the righthand side, we have $\mathcal{L}^n(A) \leq \mathcal{H}^n(A)$.

Next, $\mathcal{H}^n$ is absolutly continuous with $\mathcal{L}^n$. We can divide a set A into dyadic cubes Q_j. The Lebesgue measure on A is a sum of these cubes. The covering cubes are a specific kind of covering set. Each cube has a well-defined diameter and the volume of a sphere with the same diameter is

$$V_{\text{sphere}} = \gamma_n(\text{diam } Q_j)^n = \gamma_n(\sqrt{n})^n \mathcal{L}^n(Q).$$

Since the Hausdorff set function $h^*_{n,\delta}(A)$ is taking infimum over all shapes of the covering,

$$\gamma_r h^*_{n,\delta}(A) \leq \inf \left\{ \sum_{j \geq 1} \gamma_n(\text{diam } Q_j)^n : \{Q_j\} \text{ are cubes, covering } A, \text{diam } Q_j < \delta \right\} = \gamma_n(\sqrt{n})^n \mathcal{L}^n(A).$$

Therefore, whenever $\mathcal{L}^n(A) = 0$ we have $\mathcal{H}^n(A) = 0$.

Lastly, we will show that $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$. Given any set A, we can choose to cover cubes such that they have a combined volume not too much larger than the volume of A,

$$\sum_{i \geq 1} \mathcal{L}^n(Q_i) \leq \mathcal{L}^n(A) + \varepsilon. \tag{9}$$

For each cube, there exists a collection of disjoint closed balls $\{B_{ik}\}_{k \geq 1}$ of diameter lesser than δ that can contained in the interior of Q_i, denoted by Q_i^o, and the part that is not in any balls has measure zero,

$$\mathcal{L}^n\left(Q_i \setminus \bigcup_{k \geq 1} B_{ik}\right) = \mathcal{L}^n\left(Q_i^o \setminus \bigcup_{k \geq 1} B_{ik}\right) = 0.$$

Since $\mathcal{H}^n$ is absolutly continuous with respect to $\mathcal{L}^n$,we have that $\mathcal{H}^n\Big(Q_i \setminus \bigcup_{k \geq 1} B_{ik}\Big) = 0$ as well. Using a property outer measure, we kickstart the chain of inequalities,

$$
\begin{aligned}
\gamma_r h^*_{n,\delta}(A) &\leq \sum_{i=1}^{\infty} \gamma_r h^*_{n,\delta}(Q_i) \\
\text{Balls} \rightarrow &= \sum_{i=1}^{\infty} \gamma_r h^*_{n,\delta}\left(\bigcup_{k=1}^{\infty} B^i_k\right) \\
\text{subadditivity} \rightarrow &\leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \gamma_r h^*_{r,\delta}\left(B^i_k\right) \\
\text{from previous step} \rightarrow &\leq \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \gamma_n \left(\text{diam } B_{ki}\right)^n = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \mathcal{L}^n\left(B_{ki}\right) \\
B_{ik} \text{ are disjointed} \rightarrow &= \sum_{i=1}^{\infty} \mathcal{L}^n\left(\bigcup_{k=1}^{\infty} B_{ki}\right) \\
\text{Balls fit the cubes } \mathcal{L}^n\text{-a.e.} \rightarrow &= \sum_{i=1}^{\infty} \mathcal{L}^n\left(Q_i\right) \\
\text{equation (9)} \rightarrow &\leq \mathcal{L}^n(A) + \varepsilon.
\end{aligned}
$$

By taking the limit of $\delta \to 0$, the left-hand side becomes the Hausdorff measure $\mathcal{H}^n(A)$. Taking limit $\epsilon \to 0$, we have $\mathcal{H}^n(A) \leq \mathcal{L}^n(A)$. Together with previous inequality, both measures coincide. □

In the proof, we use the isodiametric inequality; now we are proving it. To begin with, a symmetrization that leads to the isodiametric inequality is defined as follows.

Definition 2.5 (Steiner symmetrization)**.** Suppose A is a subset in $\mathbb{R}^n$, $a \in \mathbb{R}^n$ with $|a| = 1$, and P_a is a plane given by normal vector a passing through the origin; then the Steiner symmetrization of A, denoted by $S_a(A)$, is

$$
S_a(A) = \bigcup_{\substack{b \in P_a \\ A \cap L^a_b \neq \emptyset}} \left\{ b + ta : |t| \leq \frac{1}{2}\mathcal{H}^1(A \cap L^a_b) \right\} \tag{10}
$$

where L^a_b is a line passing through b with direction vector a.

Notice that one can replace $\mathcal{H}^1$ in this definition with $\mathcal{L}^1$ as we showed eariler that they conincided. Intuitively, what this symmetrization does is captured in the figure 4. There are two important properties that we use in the proof. The first property is that the Steiner symmetrization is a diameter non-increasing operation.

Lemma 2.6. Suppose $A \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$, we have

$$
\text{diam } S_a(A) \leq \text{diam } A \tag{11}
$$

Proof. For diam $A = \infty$, the inequality is trivial. For diam $A < \infty$, suppose A is closed. For a value of $\varepsilon > 0$, we can always choose $x, y \in S_a(A)$ such that the distance between x and y is ε-close to the diameter. That is for all $\varepsilon > 0$ there exists $x, y \in S_a(A)$ such that

$$
\text{diam } S_a(A) - d(x, y) \leq \varepsilon \tag{12}
$$

Next, we will choose points from L^x_a and L^y_a as follows

$$
\begin{aligned}
b &= x - (x \cdot a)a & c &= y - (y \cdot a)a \\
r &= \inf\{t \in \mathbb{R} | b + ta \in A\} & u &= \inf\{t \in \mathbb{R} | c + ta \in A\} \\
s &= \sup\{t \in \mathbb{R} | b + ta \in A\} & v &= \sup\{t \in \mathbb{R} | c + ta \in A\}
\end{aligned}
$$

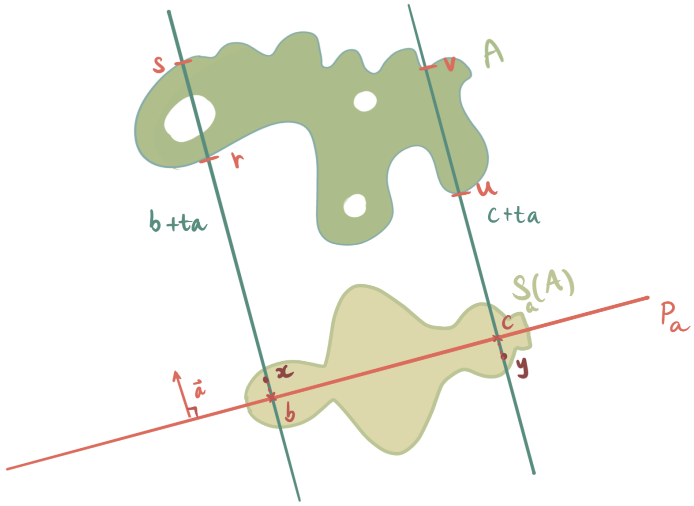


Figure 4: Choosing points in $S_a(A)$ and A. A dinosaur-shape set A and its symmetrization $S_a(A)$.

These points are illustrated in the figure 4. Suppose $v - r \geq s - u$, otherwise we swap point x and y, this leads to

$$\begin{aligned} v - r &\geq \frac{1}{2}(v-r) + \frac{1}{2}(s-u) \\ &= \frac{1}{2}(s-r) + \frac{1}{2}(v-u) \\ &\geq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a) + \frac{1}{2}\mathcal{H}^1(A \cap L_c^a). \end{aligned}$$

Since x and y are in $S_a(A)$, they satisfies the inequalities $|x \cdot a| \leq \frac{1}{2}\mathcal{H}^1(A \cap L_b^a)$ and $|y \cdot a| \leq \mathcal{H}^1(A \cap L_c^a)$. This leads to

$$v - r \geq |x \cdot a| + |y \cdot a| \geq |x \cdot a - y \cdot a| = |(x-y) \cdot a|.$$

All of these together lead to

$$\begin{aligned} (\text{diam } S_a(A) - \varepsilon)^2 &\leq d(x,y)^2 \\ &= d(b,c)^2 + |(x-y) \cdot a|^2 \\ &\leq d(b,c)^2 + (v+r)^2 \\ &= d(b + va, c + ra)^2 \\ &\leq (\text{diam } A)^2. \end{aligned}$$

For all $\varepsilon > 0$, this is hold and a diameter is always nonnegative; then diam $S_a(A) \leq$ diam A □

The second property of the Steiner symmetrization is that it conserves the Lebesgue measure.

Lemma 2.7. Suppose $A \subset \mathbb{R}^n$ Lebesgue measurable and $a \in \mathbb{R}^n$, we have

$$\mathcal{L}^n(S_a(A)) = \mathcal{L}^n(A). \tag{13}$$

Proof. Without the loss of generality, we choose $a = e_n$ as $\mathcal{L}^n$ is rotational invariant. Using the fact that $\mathcal{L}^1 = \mathcal{H}^1$ on $\mathbb{R}^1$, $f(b) = \mathcal{H}^1(A \cap L_b^a)$ is $\mathcal{L}^{n-1}$-measurable by Fubini's theorem. Also, Fubini's theorem implies the measure of A is equal to integration over cross sections of A,

$$\mathcal{L}^n(A) = \int_{\mathbb{R}^{n-1}} \mathcal{H}^1(A \cap L_b^a)db = \int_{\mathbb{R}^{n-1}} f(b)db.$$

$S_a(A)$ is $\mathcal{L}^n$-measurable because it can be rewrite as

$$S_a(A) = \left\{(b, y) : |y| \leq \frac{f(b)}{2}\right\} \setminus \left\{(b, 0) : L_b^a \cap A = \emptyset\right\}.$$

Since $S_a(A)$ has the same cross-section in a as A, then the Lebesgue measure on $\mathbb{R}^n$ is the same. ☐

Lemma 2.8 (Isodiametric inequality). Suppose A is Lebesgue measurable subsets in $\mathbb{R}^n$, then

$$\mathcal{L}^n(A) \leq \gamma_n(\text{diam } A)^n \tag{14}$$

Proof. Let diam $A < \infty$, and $\{e_1, e_2, \ldots, e_n\}$ is a standard basis of $\mathbb{R}^n$ and $A_1 = S_{e_1}(A)$, $A_2 = S_{e_2}(A_1)$, ..., $A_n = S_{e_n}(A_{n-1})$. The proof of this lemma is divided into three steps: showing that A_n is symmetric with respect to the origin, showing that the A_n satisfies the inequality, and lastly the proof of the lemma for any Lebesgue measurable subsets A.

Firstly, A_n is symmetric with respect to the origin. That is $x \in A_n$ implies $-x \in A_n$. This is straightforward as x is of the same distance from the origin as $-x$.

Secondly, to show that $\mathcal{L}^n(A_n) \leq \gamma_n(\text{diam } A_n)^n$, consider an $x \in A_n$. We have that $-x \in A_n$ as well by the first step. That is diam $A_n \geq d(x, -x) = 2|x|$. Then A_n can be covered by a ball of diameter diam A_n. Then the hypervolume of A_n is less than the hypervolume of a hypersphere of the same diameter.

For the last step, we consider a closure of A, denoted by $\overline{A}$. Using the same construction, we can have $\overline{A}_n$, which also satisfies $\mathcal{L}^n(\overline{A}_n) \leq \gamma_n(\text{diam } \overline{A}_n)^n$. Consider inequalities,

$$\begin{aligned} \mathcal{L}^n(A) &\leq \mathcal{L}^n(\overline{A}) \\ \text{(Lemma 2.7)} \rightarrow &= \mathcal{L}^n(\overline{A}_n) \\ &\leq \gamma_n(\text{diam } \overline{A}_n)^n \leq \gamma_n(\text{diam } \overline{A})^n = \gamma_n(\text{diam } A)^n. \end{aligned}$$

This completes the proof of isodiametric inequality and the proof of $\mathcal{H}^n(A) = \mathcal{L}^n(A)$. ☐

3 Hausdorff dimensions

3.1 Definition and basic properties

Definition 3.1 (Hausdorff dimension). A Hausdorff dimension of a subset S of X is defined as

$$\text{Hdim } S = \inf \{r \geq 0 : \mathcal{H}^r(S) = 0\} \tag{15}$$

If for all r, $\mathcal{H}^r(S) = \infty$, the dimension is ∞. If for all r, $\mathcal{H}^r(S) = 0$, the dimension is 0.

This definition is equivalent with

$$\text{Hdim } S = \sup \{r \geq 0 : \mathcal{H}^r(S) > 0\} \tag{16}$$

The following theorem ensures that the definition dimension respects the containment and union. The theorem is from [Nad08]. For intuition, a 3 dimension ball contains singletons (0D) and intervals (1D); and a union of the ball with a singleton should be described as a $3D$ object.

Theorem 3.2. Let A,B be Borel sets.

1. If $A \subseteq B$, then Hdim $A \leq$ Hdim B
2. Hdim $(A \cup B) = \max \{\text{Hdim } A, \text{Hdim } B\}$

Proof. The main idea for the first statement is a property of measure, $A \subseteq B \implies \mathcal{H}^r(A) \leq \mathcal{H}^r(B)$. Suppose Hdim $B = s$, then for any $\varepsilon > 0$ the measure $\mathcal{H}^{s+\varepsilon}(B) = 0$ (using equation (16)). Since $\mathcal{H}^r$ is a non-negative measure, $\mathcal{H}^{s+\varepsilon}(A) = 0$ as well. This is true for all $\varepsilon > 0$, therefore Hdim A cannot be bigger than Hdim B.

For the second statement, the $\geq$ direction is given by the first statement and the fact that A and B are contained in $A \cup B$. For the converse direction, consider s to be some value larger than $\max\{\text{Hdim } A, \text{Hdim } B\}$. The subadditivity of a measure implies

$$\mathcal{H}^s(A \cup B) \leq \mathcal{H}^s(A) + \mathcal{H}^s(B) = 0$$

This shows that $s \in \{r \geq 0 : \mathcal{H}^r(A \cup B) = 0\}$, then Hdim $(A \cup B)$ is at most s. This completes the proof. ☐

The problem with the coastline paradox is the scale dependent of of coastal lenght, the following theorem ensures that the Hausdorff dimension does not depend on the scale and the unit of measurement used. The definition and theorem are from Chapters 1 and 2, respectively, of [Nad08].

Definition 3.3 (Similarity)**.** A function $f : S \to T$ is a similarity if and only if there exists a positive number λ such that

$$d(f(x), f(y)) = \lambda d(x, y) \quad \forall x, y \in S. \tag{17}$$

Theorem 3.4. Let $f : S \to T$ be a similarity with the ratio $\lambda \geq 0$, let r be a positive real number, and let $F \subseteq S$ be a Borel set. Then $h_r^*(f[F]) = \lambda^r h_r^*(F)$ and consequently $\text{Hdim} f[F] = \text{Hdim} F$.

Proof. We will utilize the definition of h_r^* to prove this statement. First, suppose a collection of covering sets of F is denoted by $\{B_j\}$. The image of F can be covered by the set $\{f(B_j)\}$. However, the diameter of $\{f(B_j)\}$ is scaled by the ratio r. Then the sum of the diameter is

$$\sum_{j\geq 1} (\text{diam } f(B_j))^r = \sum_{j\geq 1} (\lambda \text{diam } B_j)^r.$$

Moreover, we can restrict the covering set for $f[F]$ to be only in $f[S]$. A similarity transformation is injective (because $\lambda > 0$), then it has an inverse on $f[S]$. When restricting the sets B_j to be of diameters no greater than δ, the image $f(B_j)$ has a diameter no greater than $\lambda\delta$. This leads to

$$\begin{aligned} h_s^*(f(F)) &= \lim_{\delta\to 0} \inf \left\{ \sum_{j\geq 1} (\text{diam } B_j)^r : B_j \text{ covering } f(F) \text{ and diam } B_j \leq \delta \right\} \\ \text{(injectivity)} \to &= \lim_{\delta\to 0} \inf \left\{ \sum_{j\geq 1} (\text{diam } f(B_j))^r : B_j \text{ covering } F \text{ and diam } B_j \leq \lambda^{-1}\delta \right\} \\ &= \lambda^r \lim_{\delta\to 0} \inf \left\{ \sum_{j\geq 1} (\text{diam } B_j)^r : B_j \text{ covering } F \text{ and diam } B_j \leq \lambda^{-1}\delta \right\} \\ &= \lambda^r h_s^*(F). \end{aligned}$$

Since F is Borel, then $\mathcal{H}^r$ on F and $f(F)$ are defined and they are vanishing at the same value of r. That is Hdim $F =$ Hdim $f(F)$. ☐

3.1.1 Dimension of non-fractal objects

Proposition 3.5 (a point)**.** Hausdorff dimension of a singleton is 0.

Proof. A singleton can be covered by itself which has zero diameter. Hausdorff r-dimensional measure of a singleton is then 0 for any r. Using the definition in Eq. (15), the least r is 0. ☐

Proposition 3.6 (a line)**.** Hausdorff dimension of an interval is 1.

Proof. Suppose an interval is $I = [a, b]$ with $a < b$. The diameter of this set is $b-a$. A Hausdorff 1-dimensional measure agrees with the Lebesgue measure and, hence has a finite $b - a$ measure by both measures. For a higher dimension $r > 1$ and a $\delta < 1$, the covering of I can be taken to be

$$I \subseteq \bigcup_{j=1}^{n} [a + (j-1)\delta, a + j\delta] =: C \tag{18}$$

where $n = \lceil \frac{b-a}{\delta} \rceil$. The sum of the diameter of the covering set is no lesser than the infimum of that over all possible covering sets. The sum is

$$\sum_{j\in[n]} \delta^r = \left\lceil \frac{b-a}{\delta} \right\rceil \delta^r \leq \left(1 + \frac{b-a}{\delta}\right)\delta^r. \tag{19}$$

Taking the limit $\delta \to 0$,

$$\begin{aligned} \lim_{\delta\to 0} \sum_{j\in[n]} \delta^r &\leq \lim_{\delta\to 0} \left(1 + \frac{b-a}{\delta}\right)\delta^r \\ &= \lim_{\delta\to 0} \delta^r + (b-a)\delta^{r-1} = 0. \end{aligned}$$

This force $\mathcal{H}^r = 0$ whenever $r > 1$. From Eq. (16), the Hausdorff dimension of an interval is 1. ☐

Note that the Hausdorff dimension of a union of intervals and a union of singletons are not necessarily equal to 0 or 1. The example of Cantor's set illustrates this fact.

3.1.2 Cantor set

The Cantor set is constructed from an interval of a unit length $[0, 1]$, and removing the middle chunk of length $\frac{1}{3}$, repeat the process by removing middle chunks of length $\frac{1}{3}$ of the remaining intervals. At each step,

Figure 5: The construction of the cantor set. The image is from Wikimedia and is in the public domain.

j, the union of the remaining intervals are called K_j a prefractal. The initial interval $[0, 1]$ is denoted by K_0. We now have a descending chain of subsets

$$K_0 \supset K_1 \supset K_2 \supset \cdots \searrow K. \tag{20}$$

The limit of this construction is K called a Cantor middle third set. The set is measurable as it's a countable subtraction of measurable sets. This set has a non-integer Hausdorff dimension.

Proposition 3.7. Hausdorff dimension of a Cantor middle third set is equal to $\log 2/\log 3$

Proof. Consider k-th prefractal of the Cantor set, the prefractal covered the fractal K with 2^k intervals, each with diameter $1/3^k$. The sum of each diameter to the power of r is as follows

$$\sum_{j=1}^{2^k} (\text{diam } B_j)^r = 2^k \left(\frac{1}{3}\right)^r.$$

The Hausdorff r-dimensional measure of K is a scaling of h_r^*,

$$\mathcal{H}^r(K) \propto \lim_{\delta \to 0} \inf \left\{ \sum_{j \geq 1} (\text{diam } B_j)^r, B_j \text{ covering } K, \text{diam } B_j < \delta \right\}$$

As $k \to \infty$, the covering set gets smaller, small enough to have a diameter lesser than any $\delta > 0$,

$$\lim_{k \to \infty} 2^k \left(\frac{1}{3}\right)^r = \lim_{k \to \infty} \left(\frac{2}{3^r}\right)^k$$

Notice that if inside the parenthesis is greater than 1 the limit diverges to ∞, if it is lesser than 1, the measure vanishes. The supremum of r such that $\mathcal{H}^r(K) > 0$ is r that makes $\frac{2}{3^r} = 1$. $r = \log 2 / \log 3 \approx 0.631$. $\square$

3.1.3 Koch curve

A Koch curve is an example of a fractal curve. Its topological dimension is 1 but its Hausdorff dimension is strictly greater. The construction and illustration of the Koch curve are previously shown in figure 2.

Proposition 3.8. The Koch curve has a Hausdorff dimension equal to $\log(4)/\log(3)$.

Proof. Let's denote j-th prefractal of the Koch curve with K_j. At the first prefractal, K_1, the covering sets can be an interval. However, unlike the Cantor set, when stepping to the next prefractal K_2, the same covering can no longer cover the added spikes. Instead, a triangle of height equals to $\sqrt{3}/6$ times its base

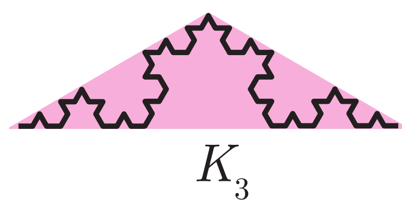


Figure 6: K_3 can be covered by an isosceles triangle with base = 1, height = $(1/3)\sin(\pi/3)$

can be used to cover the set K. The diameter of a triangle is equal to its base. For the $\delta \geq 1$, we can use this to cover the set. For a delta in range $\delta \leq \frac{1}{3^{k-1}}$, we can use the triangles with base $\frac{1}{3^k}$ to cover the set K. The number of covering is 4^k. The sum of the diameter is then

$$\sum_{j=1}^{4^k} \left(\frac{1}{3^k}\right)^r = 4^k \left(\frac{1}{3^k}\right)^r = \left(\frac{4}{3^r}\right)^k.$$

When taking limit $\delta \to 0$, it is taking limit $k \to \infty$, the limit is finite only when $r = \log 4 / \log 3$. $\square$

3.1.4 Sierpinski gasket

The Sierpinski gasket is another example of a fractal shape with a dimension greater than 1 while its topological dimension is 1.

Proposition 3.9. The Sierpinski gasket has a Hausdorff dimension of $\log 3 / \log 2$

We will not write out the proof of this proposition. The proof is similar to the proof of the Cantor set. Begin with covering with the first prefractal and as limit $\delta \to 0$ pick the further prefractal to be the covering sets.

Figure 7: The first five prefractal of the Sierpinski gasket. The image is taken from Wikimedia and is in the public domain.

3.2 Comparisions to similarity dimension

Definition 3.10 (Similarity dimension)**.**

$$\text{Sdim } S = -\frac{\log(k)}{\log(\lambda)} \tag{21}$$

where $k \geq 2$ is an integer and $\lambda < 1$ is a real number such that there is a disjoint union $S = S_1 \cup ... \cup S_k$ that for each j, $\lambda S := \{\lambda x : x \in S\}$ is congruent to S_j.

For a fractal that is self-similar (not any fractals), one can use its self-similarity to describe its fractal dimension. Intuitively, k is a number of copies of itself, and λ is a contraction factor. For example, the Koch curve has four copies $k = 4$ of itself each with a size one-third of the original $\lambda = 1/3$. We can also say that the Koch curve has sixteen copies of itself each with a size one-ninth of the original. Both result in the same similarity dimension. The Hausdorff dimension of the same fractal need not be the same as its similarity

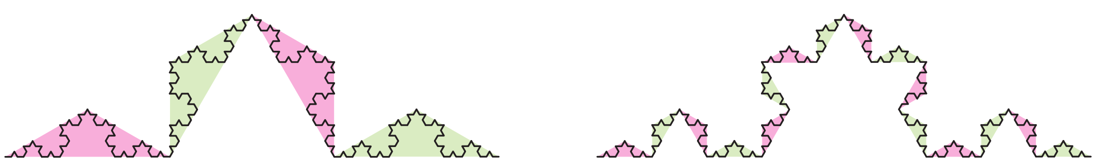


Figure 8: Two of many possible ways to count number copies. The left figure counts 4 copies, each with a color either green or magenta. While the right figure counts 16 copies, but with a smaller size.

dimension. However, there are some useful relations: the similarity dimension provides an upper bound for the Hausdorff dimension and the finite non-zero Hausdorff measure implies the similarity dimension. The definition of similarity dimension and the following theorem are from chapter 12 of [Tay06].

Proposition 3.11. $S \subset \mathbb{R}^n$ is self-similar, then

$$r > \text{Sdim } S \implies \mathcal{H}^r(S) = 0, \tag{22}$$

provided $h^*_{r,\delta} < \infty$ for some $\delta > 0$. Furthermore, for any $r \geq 0$,

$$0 < \mathcal{H}^r(S) < \infty \implies r = \text{Sdim } S \tag{23}$$

Proof. Let us denote a collection of covering sets of S as $\{B_j\}_{j=1}^{\infty}$. Since S is self-similar, suppose it has k copies of itself with contraction ratio λ. For each, B_j perform k similarity transformations of contraction ratio λ on B_j to get $\{B_{jl}\}_{l=1}^{k}$. Then the diameter of the covering sets satisfy

$$\sum_{j,l} (\text{diam } B_{jl})^r = k\lambda^r \sum_j (\text{diam } B_j)^r.$$

This means we can take B_j that are coverings of size no larger than δ, perform the similarity transformations and the diameters will be no greater than $\delta\lambda^r$. This implies

$$\left\{k\lambda^r \sum_j (\text{diam } (B_j))^r : \text{diam } B_j \leq \delta\right\} \subseteq \left\{\sum_{j,l\geq 1} (\text{diam } (B_{jl}))^r : \text{diam } B_{jl} \leq \delta\lambda\right\}$$
$$\text{and} \quad k\lambda^r h^*_{r,\delta}(S) \geq h^*_{r,\delta\lambda}(S).$$

As $\delta \to 0$, we have that k the number of copies of size lesser than δ grows while λ shrinks. Since $r >$ Sdim $S = -\log_\lambda(k)$, we have $k\lambda^r < 1$. Therefore $\mathcal{H}^r(S) = 0$

For the second statment, we use $\mathcal{H}^r(\lambda B_j) = \lambda^r \mathcal{H}^r(B_j)$. Since S is self similar with k copies of contraction ratio λ, we have that $k\lambda^r\mathcal{H}^r(S) = \mathcal{H}^r(S)$. The premise is that $\mathcal{H}^r(S)$ is finite. We then have $k = \lambda^{-r}$, equivalently $r = -\log_\lambda(k) =$ Sdim . □

Self-similarity need not to be coincide with the Hausdorff dimension. Moreover, Mandelbrot's characterization of fractals using the Hausdorff dimension cannot be rephrased directly using the similarity dimension. Some sets are fractals but their similarity dimension is the same as their topological dimension. For example, a set of rationals in $[0, 1)$

Proposition 3.12. Let's $\mathcal{Q} = [0, 1) \cap \mathbb{Q}$. Then,

$$\text{Hdim } \mathcal{Q} = 0 \quad \text{while} \quad \text{Sdim } \mathcal{Q} = 1.$$

Proof. $p \in [0, 1/2) \cap \mathbb{Q}$ can be bijectively mapped to $q \in [0, 1)$ by $q = 2p$, similarly for $p \in [1/2, 1) \cap \mathbb{Q}$. Using $\mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2$ where $\mathcal{Q}_1 = [0, 1/2) \cap \mathbb{Q}$ and $\mathcal{Q}_2 = [1/2, 1) \cap \mathbb{Q}$, this means that $k = 2$ and $\lambda = 1/2$. Therefore, a similarity dimension of $\mathcal{Q}$ is 1. For the Hausdorff dimension, $\mathcal{Q}$ can be covered with a set of countable singletons, which have no diameter. Using the definition in equation (15), the dimension is 0. □

4 Conclusion

A Hausdorff r-dimensional measure of a Borel set A, $\mathcal{H}^r(A)$, is defined to be (1.) proportional to the infimum of the sum of diameters exponentiated by r of infinitesimally small covering sets and (2.) coincide with Lebesgue measure when r is an integer.

A Hausdorff dimension extends the concept of dimension to real numbers rather than just integers. For any Borel sets A, its dimension is given by the supremum of r such that $H^r(A) > 0$ or equivalently by the infimum of r such that $H^r(A) = 0$. This is similar to the fact that the square has no volume and a line has no area.

For fractals with self-similarity, another fractal dimension called similarity dimension can be defined. It gives a lower bound for the Hausdorff dimension. However, similarity dimension of a fractal object does not necessarily coincide with its Hausdorff dimension.

4.1 Fun stuffs

- Hunting for hidden details in fractals with this website[2]
- Youtube video: Sounds of the Mandelbrot Set [3]
- Canada ranked no.1 in coastline lenght[4] (measureing at scale 1:250,000).

[2] https://mandelbrotandco.com/en.hub169.html?set=BurningShip
[3] https://www.youtube.com/watch?v=GiAj9WW1OfQ
[4] https://en.wikipedia.org/wiki/List_of_countries_by_length_of_coastline

References

[EG15] L. C. Evans and R. Gariepy. "Measure theory and fine properties of functions, revised edition". In: (2015). DOI: 10.1201/b18333.

[Man82] B. B. Mandelbrot. *The fractal geometry of nature.* eng. Updated and augm. New York: Freeman, 1983 - 1982. ISBN: 0716711869.

[Nad08] F. Nadel. "Measure, topology, and fractal geometry". In: *Undergraduate Texts in Mathematics* (2008). DOI: 10.1007/978-0-387-74749-1.

[Nat12] C. Natoli. "Fractal as fixed points of itereated function systems". 2012.

[Tay06] M. E. Taylor. *Measure theory and integration.* eng. Graduate studies in mathematics ; v. 76. Providence, R.I: American Mathematical Society, 2006. ISBN: 0821841807.

[Wei] E. Weisstein. "Hypersphere". In: *From MathWorld–A Wolfram Web Resource* (). URL: https://mathworld.wolfram.com/Hypersphere.html.